Sasa-Satsuma higher-order nonlinear Schrödinger equation and its bilinearization and multisoliton solutions

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Higher-order and multicomponent generalizations of the nonlinear Schrödinger equation are important in various applications, e.g., in optics. One of these equations, the integrable Sasa-Satsuma equation, has particularly interesting soliton solutions. Unfortunately, the construction of multisoliton solutions to this equation presents difficulties due to its complicated bilinearization. We discuss briefly some previous attempts and then give the correct bilinearization based on the interpretation of the Sasa-Satsuma equation as a reduction of the three-component Kadomtsev-Petviashvili hierarchy. In the process, we also get bilinearizations and multisoliton formulas for a two-component generalization of the Sasa-Satsuma equation (the Yajima-Oikawa-Tasgal-Potasek model), and for a $(2+1)$-dimensional generalization.

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I. INTRODUCTION

One of the most interesting applications of solitons is in the propagation of short pulses in optical fibers (for an overview, see Ref. [1]). The basic phenomena are described by the nonlinear Schrödinger equation, but as the pulses get shorter various additional effects become important. In Ref. [2] Kodama and Hasegawa derived the relevant equation with higher-order correction terms, the generic form of such an equation is (in the optical fiber setting the roles of time and space are usually reversed)

$$iq_\xi+\alpha_1 q_{\tau\tau}+\alpha_2|q|^2q+i[\beta_1 q_{\tau\tau\tau}+\beta_2|q|^2q_\tau+\beta_3 q(|q|^2)_\tau]=0, \tag{1}$$

where the α_i,β_i are real constants and q a complex function. The first three terms form the standard nonlinear Schrödinger equation (NLS) and the β_i terms are the perturbative corrections. Usually, one chooses the scaling so that $\alpha_2=2\alpha_1$. In this paper, we assume $\beta_1\neq 0$.

Our main concern here is the bilinearization and multisoliton solutions of the Sasa-Satsuma equation (SSNLS) [3], which is a particularly interesting integrable example in the above class. In this section, we discuss some basic properties of Eq. (1), its integrable special cases and their multicomponent generalizations. In particular, we show that many previous attempts to solve these equations have produced only rather trivial solutions, in which the complex and multicomponent freedom has been "frozen." The reason for this turns out to be in the incorrect bilinearization that was used in those papers. The correct bilinearization (presented in Sec. II with detailed derivation in Sec. III) follows once we identify SSNLS as a reduction of the three-component Kadomtsev-Petviashvili (KP) hierarchy, and then we also obtain general multisoliton solutions.

A. Gauge transformation

In order to understand the complex structure of Eq. (1), it is important to isolate the gauge (phase) invariance and fix the gauge. First, let us recall that the NLS part of Eq. (1) (i.e., if $\beta_i=0$) is invariant under the combined gauge-Galilei transformation

$$q(\xi,\tau)=e^{iv(\tau-v\xi)/\alpha_1}y(x,t),\quad x=\tau-2v\xi,\quad t=\xi. \tag{2}$$

The full equation (1) is not invariant under Eq. (2), but if we try the transformation

$$q(\xi,\tau)=e^{i(c_1\tau+c_2\xi)}y(x,t),\quad x=\tau+c_3\xi, t=\xi,$$
$$c_i \text{ real constants}, \tag{3}$$

we find that if

$$c_3=c_1(-2\alpha_1+3\beta_1c_1),\quad c_2=c_1^2(-\alpha_1+c_1\beta_1), \tag{4}$$

then Eq. (1) is *form invariant*: the equation for $y(x,t)$ is as in Eq. (1) with β_i unchanged, but the α_i change according to

$$\alpha_1\to\tilde{\alpha}_1=\alpha_1-3\beta_1c_1,\quad \alpha_2\to\tilde{\alpha}_2=\alpha_2-\beta_2c_1. \tag{5}$$

We can therefore use this transformation to put $\alpha_1=\alpha_2=0$, provided that

$$3\beta_1\alpha_2=\beta_2\alpha_1. \tag{6}$$

[In the usual normalization $\alpha_1=\frac{1}{2}$, $\alpha_2=1$ (6) means $\beta_2=6\beta_1$.] In all integrable cases (along with some nonintegrable cases appearing in the literature) Eq. (6) is satisfied, and we assume it from now on.

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On the basis of the above, *we fix gauge* (3) *by requiring that* $\alpha_i=0$ *in the equation* and compare results only in that uniquely defined gauge.

B. Integrable cases

The integrability of the class of equations (1) has been studied by a number of authors using Painlevé analysis [4–6] and other methods [7], with the consistent result that if $\beta_1,\beta_2\neq 0$ there are precisely two integrable cases with bright solitons.

(1) Hirota's equation (HNLS) [8]: $\beta_1:\beta_2:\beta_3=1:6:3$,

$$q_t+q_{xxx}+6|q|^2q_x=0, \tag{7}$$

(2) Sasa-Satsuma equation (SSNLS) [3]: $\beta_1:\beta_2:\beta_3=1:6:3$,

$$q_t+q_{xxx}+6|q|^2q_x+3q|q^2|_x=0. \tag{8}$$

Here, the scaling convention mentioned above has been assumed and the α_i terms eliminated.

Some nonintegrable special cases of Eq. (1) have also been studied in the literature, including [9,10]: $\beta_1:\beta_2:\beta_3=1:6:6$,

$$q_t+q_{xxx}+6(q|q^2|)_x=0. \tag{9}$$

C. Multicomponent generalizations

Both HNLS and SSNLS allow various kinds of multicomponent generalizations, some of them integrable. The results of a Painlevé analysis [11] can be summarized as follows

Case 1:

$$q_{1t}+q_{1xxx}+3(|q_1|^2+|q_2|^2)q_{1x}=0,$$

$$q_{2t}+q_{2xxx}+3(|q_1|^2+|q_2|^2)q_{2x}=0, \tag{10}$$

which can be interpreted as a real four-component modified Korteweg–de Vries (mKdV) equation, reducing to HNLS for $q_1=q_2$, etc.;

Case 2 [11]:

$$q_{1t}+q_{1xxx}+3(|q_1|^2+|q_2|^2)q_{1x}+3q_1(|q_2|^2)_x=0,$$

$$q_{2t}+q_{2xxx}+3(|q_1|^2+|q_2|^2)q_{2x}+3q_2(|q_1|^2)_x=0; \tag{11}$$

and Case 3 [12]:

$$q_{1t}+q_{1xxx}+3(|q_1|^2+|q_2|^2)q_{1x}+\tfrac{3}{2}q_1(|q_1|^2+|q_2|^2)_x=0,$$

$$q_{2t}+q_{2xxx}+3(|q_1|^2+|q_2|^2)q_{2x}+\tfrac{3}{2}q_2(|q_1|^2+|q_2|^2)_x=0, \tag{12}$$

which reduce to SSNLS under the above reduction; and a mixed case,

Case 4 [13,14]:

$$q_{1t}+q_{1xxx}+\frac{a}{2}(|q_1|^2+|q_2|^2)q_{1x}+\frac{a}{2}q_1(q_1^*q_{1x}+q_2^*q_{2x})=0,$$

$$q_{2t}+q_{2xxx}+\frac{a}{2}(|q_1|^2+|q_2|^2)q_{2x}+\frac{a}{2}q_2(q_1^*q_{1x}+q_2^*q_{2x})=0, \tag{13}$$

which reduces to HNLS under $q=q_1=q_2, a=3$ and to SSNLS under $q=q_1=q_2^*, a=6$. In each case, we must, of course, adjoin the complex conjugated equations. Cases 1–3 are invariant under $q_2\leftrightarrow q_2^*$, while case 4 changes to the alternative form

Case 4′:

$$q_{1t}+q_{1xxx}+\frac{a}{2}(|q_1|^2+|q_2|^2)q_{1x}+\frac{a}{2}q_1(q_1^*q_{1x}+q_2q_{2x}^*)=0,$$

$$q_{2t}+q_{2xxx}+\frac{a}{2}(|q_1|^2+|q_2|^2)q_{2x}+\frac{a}{2}q_2(q_1q_{1x}^*+q_2^*q_{2x})=0. \tag{14}$$

Under the reduction $q_2=0$, cases 1,2,4 reduce to HNLS and 3 to SSNLS.

D. The modified Korteweg–de Vries limit

With complex and multicomponent equations, it is important to make the following observation: we can always make the real, one-component reduction

$$q_i(x,t)=c_iu(x,t) \quad \forall i, \tag{15}$$

where u is a *real* function and c_i are arbitrary complex constants. As a result of this, *all* the equations mentioned before (and many others, including nonintegrable ones) reduce to the real mKdV equation

$$u_t=u_{xxx}+\kappa u^2u_x. \tag{16}$$

(Note that for case 2, we need $|c_1|=|c_2|$.) This was observed already in Ref. [4]; see Eqs. (21)–(25). A consequence of this rather simple observation is the following.

All these complex and/or multicomponent systems always have multisoliton solutions of the real mKdV type, with frozen complex and/or multicomponent freedom.

In the usual real one-component case, the existence of multisoliton solutions is a sign of integrability, but from the above we can see that this is not necessarily true in the complex or multicomponent case. In general, it is essential that the individual solitons, from which the multisoliton solution is built, are each allowed to have their own freedom of initial position and overall phase. That is, even if a one-soliton solution is of type (15), in the multisoliton case each component soliton must be allowed to have its own parameters, including the complex coefficient(s) c_i. Furthermore, during scattering some of these parameters can change [15].

Thus, in practice reduction (15) trivializes the equation and the resulting solutions are hardly of interest. Nevertheless, it seems that several recent studies have fallen into this trap and produced no solutions with genuine multicomponent or complex structure. This is quite evident from the proposed final results: for example, the multicomponent structure is trivialized into a constant factor in Ref. [16] [see Eqs. (3.15), (3.16) or (3.20), or (3.25), (3.26), (3.32), (3.33)] [10] [see Eq. (17) or (24) or (27)], and [17] [see Eq. (10) or (13)], whereas the solutions are obviously real (after the gauge has been fixed) in Ref. [9] (see Sec. III), [18] [see Eqs. (2), (3), (15), and (16)], and [19] (in Sec. IV k_i, η_i are real and H/G a constant). Below, we will show that the reason for this often lies in the incorrect bilinearization that was used.

E. Traveling-wave solutions

Let us now return to the one-component Eqs. (7) and (8) and consider their one-soliton solutions. For the purpose of orientation, let us first consider HNLS (7). The usual traveling-wave ansatz

$$q(x,t)=e^{i(ax+bt+\omega)}f(x+dt+\delta), \tag{17}$$

where f is a real function (soliton envelope), leads to a pair of real equations, which are compatible, if

$$b=a(3d-8a^2), \tag{18}$$

and in that case the solution can be parametrized as follows:

$$q(x,t)=e^{ia[x+(a^2-3c^2)t+\omega]}\frac{c}{\cosh[c\{x+(3a^2-c^2)t+\delta\}]}. \tag{19}$$

We observe that there are four free real parameters: a and c, which relate to the size and velocity of the soliton, and ω,δ which give the constant complex phase and soliton position, respectively.

If we use the same ansatz (17) in Eq. (8), we find that it works only under the additional condition $a=0$, leading to

$$q(x,t)=\frac{ce^{i\omega}}{\sqrt{2}\cosh[c(x-c^2t+\delta)]}. \tag{20}$$

Since one parameter was lost, solution (20) is not general enough. Indeed, Sasa and Satsuma have derived a complex traveling-wave solution to Eq. (8), which does not fit the usual ansatz (17) but has the form [3]

$$q(x,t)=e^{ia[x+(a^2-3c^2)t+\omega]}\frac{2e^{\eta}c(e^{2\eta}+\kappa)}{e^{4\eta}+2e^{2\eta}+|\kappa|^2},$$

$$\kappa=\frac{a}{a+ic},\quad \eta=c[x+(3a^2-c^2)t+\delta]. \tag{21}$$

We note that this has similar x,t dependence as Eq. (19) but the functional form is different; also in the limit $a\to 0$, i.e., $\kappa\to 0$, we obtain the real limit (20).

It turns out that Eq. (21) is still not the most general one-soliton solution for this system; it is given by $q=G/F$, where

$$G=\gamma e^{\eta}+\rho^* e^{\eta^*}+m\left(\frac{\gamma}{2p^2}e^{2\eta+\eta^*}+\frac{\rho^*}{2p^{*2}}e^{\eta+2\eta^*}\right), \tag{22}$$

$$\begin{aligned}F&=1+2\frac{|\rho|^2+|\gamma|^2}{(p+p^*)^2}e^{\eta+\eta^*}+\frac{\rho\gamma}{2p^2}e^{2\eta}+\frac{\rho^*\gamma^*}{2p^{*2}}e^{2\eta^*}\\&\quad+\frac{|m|^2}{4|p|^4}e^{2(\eta+\eta^*)}\\&=1+\tfrac{1}{2}\left|\frac{\gamma e^{\eta}}{p}+\frac{\rho^* e^{\eta^*}}{p^*}\right|^2\\&\quad+\tfrac{1}{2}(|\gamma|^2+|\rho|^2)\left|\frac{(p-p^*)e^{\eta}}{(p+p^*)p}\right|^2+\left|\frac{me^{2\eta}}{2p^2}\right|^2,\end{aligned} \tag{23}$$

$$m=(|\gamma|^2p-|\rho|^2p^*)\frac{p-p^*}{(p+p^*)^2}, \tag{24}$$

$$\eta=px-p^3t+\eta^{(0)},\quad p,\ \rho,\ \gamma,\ \text{and}\ \ \eta^{(0)}\ \ \text{complex}. \tag{25}$$

Comparing with the original one-soliton solution (21), we have two extra parameters γ and ρ. By η translation one finds that only ρ/γ matters, and if it vanishes we have the usual SS solution, so this is a genuine new parameter. This parameter controls the oscillation, which appears not only in the carrier but also in the envelope (but in any case $F\geqslant 1$ so the solution is not singular). This solution was already obtained by Mihalache *et al.* [20] using inverse scattering transform, below we will derive it using the bilinear method. It is not easy to derive such a solution from a (complex) traveling-wave ansatz, and Hirota's bilinear method is easier to use in this case.

F. Outline

In this paper, we first present in Sec. II the bilinearizations that work and the one-soliton solution that is obtained by the expansion method. The detailed derivations and multisolution solutions are made in Sec. III.

It is well known that soliton equations can be organized into infinite hierarchies as described by the Sato theory [21] and that particular equations can be obtained from these hierarchies by various reductions. Indeed, one cannot have a full understanding of an integrable equation before its relation to integrable hierarchies is described. In Sec. III, we will give the full picture by showing that the Sasa-Satsuma equation can be obtained from the general Sato theory as a reduction of the three-component KP hierarchy. The reduction can be made in two different ways producing two different bilinearizations. As intermediate steps of the reduction process, we get either a $(2+1)$-dimensional generalization or a complex two-component generalization of the Sasa-Satsuma equation.

II. DIRECT BILINEARIZATION AND ONE-SOLITON SOLUTIONS

One can attempt to bilinearize the generic equation

$$q_t+q_{xxx}+6|q|^2q_x+\beta q(|q|^2)_x=0 \tag{26}$$

with the standard substitution

$$q=\frac{G}{F}, \tag{27}$$

where F is taken to be real and G complex. This leads to the equation

$$F^2[(D_x^3+D_t)G\cdot F]-\beta GF(D_xG\cdot G^*)-3(D_xG\cdot F)\times[D_x^2F\cdot F-\tfrac{2}{3}(\beta+3)|G|^2]=0, \tag{28}$$

which is quartic in F,G. Here D_x and D_t are the Hirota bilinear operators. We can see that if $\beta=0$ (which is the HNLS case) the equation splits naturally into two bilinear ones, $(D_x^3+D_t)G\cdot F=0$ and $D_x^2F\cdot F=2|G|^2$. In the general case (that includes the SSNLS equation at $\beta=3$), we could take

$$D_x^2F\cdot F=\tfrac{2}{3}(\beta+3)|G|^2, \tag{29}$$

as one of the equations, which leaves a trilinear equation

$$F[(D_x^3+D_t)G\cdot F]-\beta G[D_xG\cdot G^*]=0. \tag{30}$$

One might be tempted to require that in Eq. (30) the terms in square brackets vanish separately, but this is not correct (as was already noted in Ref. [4]) because it would result in more independent equations than there are unknowns and in effect force reduction to the real mKdV equation. [Clearly $D_xG\cdot G^*=0\Leftrightarrow\partial_x(G/G^*)=0$ and therefore, the phase of G has no x dependence, and when $G=R(x,t)e^{i\theta(t)}$ is substituted into the remaining equation one finds that $\theta(t)$ must be constant, i.e., Eq. (15).] As a matter of fact, this sort of *brute force bilinearization* turns out to be precisely the reason for the trivialization of the complex multicomponent freedom mentioned before. Unfortunately, this incorrect approach has been used quite frequently, see, e.g., Eq. (3.6) of Ref. [16]; Eqs. (9), (29), (44), and (47) of Ref. [10]; Eq. (12) of Ref. [22]; Eq. (26) of Ref. [23]; Eqs. (21)–(24) of Ref. [24]; Eq. (7c) of Ref. [18]; Eq. (20) of Ref. [19]; Eq. (43) of Ref. [25]; Eq. (18) of Ref. [26]; Eqs. (9) and (16) of Ref. [17]; Eq. (38) of Ref. [27].

The trilinear equation (30) can only be split into two bilinear ones by introducing a new dependent variable. There are two acceptable ways to do this, resulting in

$$\begin{aligned}
&D_x^2F\cdot F=\tfrac{2}{3}(\beta+3)|G|^2,\\
&[(6-\beta)D_x^3+2(\beta+3)D_t]G\cdot F=3\beta D_xH\cdot F,\\
&[(6-\beta)D_x^3+2(\beta+3)D_t]G^*\cdot F=3\beta D_xH^*\cdot F,\\
&D_x^2G\cdot F=-HF,\\
&D_x^2G^*F\cdot=-H^*F,
\end{aligned} \tag{31}$$

or

$$\begin{aligned}
&D_x^2F\cdot F=\tfrac{2}{3}(\beta+3)|G|^2,\\
&(D_x^3+D_t)G\cdot F=\beta SG,\\
&(D_x^3+D_t)G^*\cdot F=-\beta SG^*,\\
&D_xG\cdot G^*=SF,
\end{aligned} \tag{32}$$

where the new dependent variable has been called H and S, respectively. Note that S is pure imaginary, H complex, and that $HG^*-H^*G=D_xF\cdot S$. These splittings are acceptable, because they introduce equal numbers of new functions and new equations and, furthermore, for integrable equations and soliton solutions the new functions turn out to be expressible in terms of polynomials of exponentials. Thus, for any β we can give for Eq. (26) a bilinear form in terms of three bilinear equations for three functions, but it should be emphasized that the fact that an equation can be written in a bilinear form does not by itself imply that the equation is integrable, or that the new functions S,H are τ functions, although it is the case when $\beta=3$.

The one-soliton solution can be obtained as usual by substituting the expansions

$$F=1+\epsilon^2F_2+\epsilon^4F_4+\cdots,\quad G=\epsilon G_1+\epsilon^3G_3+\cdots \tag{33}$$

accompanied by suitable ansatz H or S, into Eq. (31) or (32), and truncating at some power of the formal expansion parameter ϵ. For HNLS $(\beta=0)$, the expansion can be truncated by keeping terms up to ϵ^2, but for SSNLS $(\beta=3)$, we must go up to ϵ^4 obtaining F,G as given in Eqs. (22)–(25), with the auxiliary functions

$$S=(p-p^*)(|\gamma|^2-|\rho|^2)e^{\eta+\eta^*}, \tag{34}$$

$$H=-\gamma p^2e^{\eta}-\rho^*p^{*2}e^{\eta^*}-\frac{m}{2}\left[\frac{\gamma p^{*2}}{p^2}e^{2\eta+\eta^*}+\frac{\rho^*p^2}{p^{*2}}e^{\eta+2\eta^*}\right], \tag{35}$$

which are also polynomials of exponentials. It is not known whether the expansion can be truncated for any other value of β.

III. THE SASA-SATSUMA EQUATION AS A REDUCTION OF THE THREE-COMPONENT KP HIERARCHY

We will next explain how the Sasa-Satsuma equation and its multisoliton solutions can be obtained from the three-component KP hierarchy by suitable reductions. It turns out that there are *two* different reduction routes leading to the Sasa-Satsuma equation; both are two-step reductions but the intermediate equations are different. We will first describe the starting point (three-component KP hierarchy) and then the two kinds of reductions.

A. Three-component KP hierarchy and its τ functions

In general, the three-component KP hierarchy has τ functions depending on three infinite sets of variables $\mathbf{x}=x,x_2,x_3,\ldots$, $\mathbf{y}=y,y_2,y_3,\ldots$, and $\mathbf{z}=z,z_2,z_3,\ldots$, and is defined in terms of vector "eigenfunctions" $\boldsymbol{\phi}(\mathbf{x})$, $\boldsymbol{\psi}(\mathbf{y})$, and $\boldsymbol{\chi}(\mathbf{z})$ and "adjoint eigenfunctions" $\overline{\boldsymbol{\phi}}(\mathbf{x})$, $\overline{\boldsymbol{\psi}}(\mathbf{y})$, and $\overline{\boldsymbol{\chi}}(\mathbf{z})$. We should emphasize that, at this point, these six eigenfunctions are independent of one another. In general, they are only assumed to satisfy the linear equations (for $n=2,3,\ldots$)

$$\partial_{x_n}\boldsymbol{\phi}=\partial_x^n\boldsymbol{\phi},\quad -\partial_{x_n}\overline{\boldsymbol{\phi}}=(-\partial_x)^n\overline{\boldsymbol{\phi}}, \tag{36}$$

$$\partial_{y_n}\boldsymbol{\psi}=\partial_y^n\boldsymbol{\psi},\quad -\partial_{y_n}\overline{\boldsymbol{\psi}}=(-\partial_y)^n\overline{\boldsymbol{\psi}}, \tag{37}$$

$$\partial_{z_n}\boldsymbol{\chi}=\partial_z^n\boldsymbol{\chi},\quad -\partial_{z_n}\overline{\boldsymbol{\chi}}=(-\partial_z)^n\overline{\boldsymbol{\chi}}. \tag{38}$$

Here, we consider the special case in which only dependence on x,x_2,x_3, y, and z is active and so the vectors $\boldsymbol{\phi}(x,x_2,x_3)$ and $\overline{\boldsymbol{\phi}}(x,x_2,x_3)$ satisfy

$$\partial_{x_2}\boldsymbol{\phi}=\partial_x^2\boldsymbol{\phi},\quad \partial_{x_3}\boldsymbol{\phi}=\partial_x^3\boldsymbol{\phi},\quad \partial_{x_2}\overline{\boldsymbol{\phi}}=-\partial_x^2\overline{\boldsymbol{\phi}},\quad \partial_{x_3}\overline{\boldsymbol{\phi}}=\partial_x^3\overline{\boldsymbol{\phi}}, \tag{39}$$

and $\boldsymbol{\psi}(y)$, $\boldsymbol{\chi}(z)$, $\overline{\boldsymbol{\psi}}(y)$, and $\overline{\boldsymbol{\chi}}(z)$ are arbitrary vector functions of a single variable.

A potential matrix m is defined by

$$\partial_x m=\boldsymbol{\phi}\overline{\boldsymbol{\phi}}^t,\quad \partial_y m=\boldsymbol{\psi}\overline{\boldsymbol{\psi}}^t,\quad \partial_z m=\boldsymbol{\chi}\overline{\boldsymbol{\chi}}^t, \tag{40}$$

which can be integrated to

$$m=c+\int\boldsymbol{\phi}\overline{\boldsymbol{\phi}}^t dx+\int\boldsymbol{\psi}\overline{\boldsymbol{\psi}}^t dy+\int\boldsymbol{\chi}\overline{\boldsymbol{\chi}}^t dz, \tag{41}$$

where c is a constant matrix. As a consequence of Eq. (36), we also have

$$\partial_{x_2}m=\boldsymbol{\phi}_x\overline{\boldsymbol{\phi}}^t-\boldsymbol{\phi}\overline{\boldsymbol{\phi}}_x^t,\quad \partial_{x_3}m=\boldsymbol{\phi}_{xx}\overline{\boldsymbol{\phi}}^t-\boldsymbol{\phi}_x\overline{\boldsymbol{\phi}}_x^t+\boldsymbol{\phi}\overline{\boldsymbol{\phi}}_{xx}^t, \tag{42}$$

Now define the τ functions

$$f=|m|, \tag{43}$$

$$g=\begin{vmatrix} m & \boldsymbol{\phi}\\ -\overline{\boldsymbol{\psi}}^t & 0\end{vmatrix},\quad \overline{g}=\begin{vmatrix} m & \boldsymbol{\psi}\\ -\overline{\boldsymbol{\phi}}^t & 0\end{vmatrix}, \tag{44}$$

and

$$h=\begin{vmatrix} m & \boldsymbol{\phi}\\ -\overline{\boldsymbol{\chi}}^t & 0\end{vmatrix},\quad \overline{h}=\begin{vmatrix} m & \boldsymbol{\chi}\\ -\overline{\boldsymbol{\phi}}^t & 0\end{vmatrix}. \tag{45}$$

By considering Jacobi determinantal identities involving f, g, $\overline{g}$, h, and $\overline{h}$ and their derivatives with respect to x, x_2, x_3, y, and z, one may compile a complete list of bilinear equations that are satisfied by these functions. The bilinear equations given below are the only ones from this list that will actually be used in the rest of this paper:

$$(D_x^2-D_{x_2})g\cdot f=0,\quad (D_x^2+D_{x_2})\overline{g}\cdot f=0, \tag{46}$$

$$(D_x^2-D_{x_2})h\cdot f=0,\quad (D_x^2+D_{x_2})\overline{h}\cdot f=0, \tag{47}$$

$$\begin{gathered}(D_x^3+3D_xD_{x_2}-4D_{x_3})g\cdot f=0,\\ (D_x^3-3D_xD_{x_2}-4D_{x_3})\overline{g}\cdot f=0,\end{gathered} \tag{48}$$

$$\begin{gathered}(D_x^3+3D_xD_{x_2}-4D_{x_3})h\cdot f=0,\\ (D_x^3-3D_xD_{x_2}-4D_{x_3})\overline{h}\cdot f=0,\end{gathered} \tag{49}$$

$$D_yD_xf\cdot f=-2g\overline{g},\quad D_zD_xf\cdot f=-2h\overline{h}, \tag{50}$$

$$D_y(D_x^2-D_{x_2})g\cdot f=0,\quad D_y(D_x^2+D_{x_2})\overline{g}\cdot f=0, \tag{51}$$

$$D_z(D_x^2-D_{x_2})h\cdot f=0,\quad D_z(D_x^2+D_{x_2})\overline{h}\cdot f=0. \tag{52}$$

As is typical for the multicomponent KP hierarchy, this set of equations appears to be overdetermined as it stands, having many more equations than dependent variables. But we already know that it has a rather general set of solutions given above (even containing several arbitrary functions of one variable). It turns out that there exist exactly the right number of differential relations among these equations to guarantee their compatibility. There is some freedom in choosing the primary or independent equations, one choice is (46), (47), (48a), and (50) (seven equations for five functions and two dummy independent variables). The remaining equations are consequences of these or possibly just restrict some constants of integration. As an example consider Eq. (48b). From Eq. (46a), we can determine g_{x_2}, from Eq. (48a), g_{x_3}, and from their cross derivatives, we get an equation for f. But now doing the same computation for $\overline{g}$ from Eqs. (46b) and (48b), we get the *same* equation for f and thus Eq. (48b) does not add essential information.

In order to write Eqs. (46)–(52) in nonlinear form, let us first introduce the dependent variables

$$q=\frac{g}{f},\quad \bar{q}=\frac{\bar{g}}{f},\quad r=\frac{h}{f},\quad \bar{r}=\frac{\bar{h}}{f},\quad \text{and}$$

$$\Phi=\tfrac{1}{2}(\ln f)_x,\quad \Psi=\tfrac{1}{2}(\ln f)_{x_2}. \tag{53}$$

Converting the bilinear equations into nonlinear form and eliminating dependence on the auxiliary variable x_2 one obtains

$$q_{xxx}+6q_x\Phi_x+3q(\Phi_{xx}+\Psi_x)-q_{x_3}=0,$$

$$\bar{q}_{xxx}+6\bar{q}_x\Phi_x+3\bar{q}(\Phi_{xx}-\Psi_x)-\bar{q}_{x_3}=0,$$

$$r_{xxx}+6r_x\Phi_x+3r(\Phi_{xx}+\Psi_x)-r_{x_3}=0,$$

$$\bar{r}_{xxx}+6\bar{r}_x\Phi_x+3\bar{r}(\Phi_{xx}-\Psi_x)-\bar{r}_{x_3}=0, \tag{54}$$

together with

$$\Phi_y=-\tfrac{1}{2}q\bar{q},\quad \Phi_z=-\tfrac{1}{2}r\bar{r},$$

$$\Psi_y=-\tfrac{1}{2}(q_x\bar{q}-q\bar{q}_x),\quad \Psi_z=-\tfrac{1}{2}(r_x\bar{r}-r\bar{r}_x). \tag{55}$$

Although this looks superficially like a $(3+1)$-dimensional system, it in fact describes a family of $(2+1)$-dimensional systems. The y and z dependence arises in such a way that it could be replaced by single variable corresponding to any linear combination of y and z.

In the following sections, we will describe a two-stage reduction of this system in which a calculation similar to that used above will give the Sasa-Satsuma equation.

B. First reduction, step 1

The previous set of equations contains two dummy variables, x_2 and one of y,z. In this reduction, we will eliminate the dummy variables by keeping just the leading terms in x_2 and $y-z$. We start by considering eigenfunctions and adjoint eigenfunctions possessing the symmetry

$$\bar{\boldsymbol{\phi}}(x,-x_2,x_3)=\boldsymbol{\phi}(x,x_2,x_3), \tag{56}$$

and the other eigenfunctions having pairwise identical forms:

$$\bar{\boldsymbol{\psi}}(a)=\boldsymbol{\chi}(a),\quad \bar{\boldsymbol{\chi}}(a)=\boldsymbol{\psi}(a). \tag{57}$$

This reduction may be shown to be a natural generalization of the three-component version of the C reduction described in Ref. [21].

Now we explore the consequences of this symmetry on the τ functions. Using the independent variables $y=\xi+\eta$ and $z=\xi-\eta$, symmetry (57) gives

$$\bar{\boldsymbol{\psi}}(y)=\boldsymbol{\chi}(\xi+\eta)=\boldsymbol{\chi}(z)|_{\eta\to-\eta},$$

$$\bar{\boldsymbol{\chi}}(z)=\boldsymbol{\psi}(\xi-\eta)=\boldsymbol{\psi}(y)|_{\eta\to-\eta}.$$

For the potential m, it is then easy to see that

$$m(x,-x_2,x_3,\xi,-\eta)=m^t(x,x_2,x_3,\xi,\eta), \tag{58}$$

as long as the constant matrix c in Eq. (41) is taken to be symmetric. Hence,

$$f(x,-x_2,x_3,\xi,-\eta)=f(x,x_2,x_3,\xi,\eta). \tag{59}$$

In a similar way,

$$\bar{g}(x,-x_2,x_3,\xi,-\eta)=h(x,x_2,x_3,\xi,\eta),$$

$$\bar{h}(x,-x_2,x_3,\xi,-\eta)=g(x,x_2,x_3,\xi,\eta). \tag{60}$$

Next we consider the Taylor expansions of the eigenfunctions with respect to x_2 and η and obtain

$$\boldsymbol{\phi}(x,x_2,x_3)=\boldsymbol{\phi}(x,0,x_3)+x_2\boldsymbol{\phi}_{xx}(x,0,x_3)+O(x_2^2), \tag{61}$$

while symmetry (56) gives

$$\bar{\boldsymbol{\phi}}(x,x_2,x_3)=\boldsymbol{\phi}(x,0,x_3)-x_2\boldsymbol{\phi}_{xx}(x,0,x_3)+O(x_2^2). \tag{62}$$

By a similar argument

$$\boldsymbol{\psi}(y)=\boldsymbol{\psi}(\xi)+O(\eta),\quad \boldsymbol{\chi}(z)=\boldsymbol{\chi}(\xi)+O(\eta),$$

$$\bar{\boldsymbol{\psi}}(y)=\boldsymbol{\chi}(\xi)+O(\eta),\quad \bar{\boldsymbol{\chi}}(z)=\boldsymbol{\psi}(\xi)+O(\eta). \tag{63}$$

For the potential m, the expansion is

$$m(x,x_2,x_3,\xi,\eta)=m(x,0,x_3,\xi,0)+x_2(\boldsymbol{\phi}_x\boldsymbol{\phi}^t-\boldsymbol{\phi}\boldsymbol{\phi}_x^t)(x,0,x_3)+O(x_2^2,\eta). \tag{64}$$

For the τ functions, we then get

$$f=f_0(x,x_3,\xi)+O(x_2^2,\eta x_2,\eta^2) \tag{65}$$

and

$$f_0=|m_0|, \tag{66}$$

where $m_0=m(x,0,x_3,\xi,0)$ satisfies

$$m_{0,x}=\boldsymbol{\phi}_0\boldsymbol{\phi}_0^t,\quad m_{0,x_3}=\boldsymbol{\phi}_{0,xx}\boldsymbol{\phi}_0^t-\boldsymbol{\phi}_{0,x}\boldsymbol{\phi}_{0,x}^t+\boldsymbol{\phi}_0\boldsymbol{\phi}_{0,x}^t,$$

$$m_{0,\xi}=\boldsymbol{\psi}\boldsymbol{\chi}^t+\boldsymbol{\chi}\boldsymbol{\psi}^t, \tag{67}$$

and $\boldsymbol{\phi}_0=\boldsymbol{\phi}(x,0,x_3)$ satisfies the single linear partial differential equation

$$\partial_{x_3}\boldsymbol{\phi}_0=\partial_x^3\boldsymbol{\phi}_0. \tag{68}$$

Also,

$$g=g_0+x_2g_2+O(x_2^2,\eta),\quad h=h_0+x_2h_2+O(x_2^2,\eta), \tag{69}$$

where

$$g_0=\begin{vmatrix} m_0 & \boldsymbol{\phi}_0 \\ -\boldsymbol{\chi}^t & 0 \end{vmatrix},\quad h_0=\begin{vmatrix} m_0 & \boldsymbol{\phi}_0 \\ -\boldsymbol{\psi}^t & 0 \end{vmatrix} \tag{70}$$

and

$$g_2=\begin{vmatrix} m_0 & \boldsymbol{\phi}_{0,xx} \\ -\boldsymbol{\chi}^t & 0 \end{vmatrix} - \begin{vmatrix} m_0 & \boldsymbol{\phi}_0 & \boldsymbol{\phi}_{0,x} \\ -\boldsymbol{\phi}_0^t & 0 & 0 \\ -\boldsymbol{\chi}^t & 0 & 0 \end{vmatrix},$$

$$h_2=\begin{vmatrix} m_0 & \boldsymbol{\phi}_{0,xx} \\ -\boldsymbol{\psi}^t & 0 \end{vmatrix} - \begin{vmatrix} m_0 & \boldsymbol{\phi}_0 & \boldsymbol{\phi}_{0,x} \\ -\boldsymbol{\phi}_0^t & 0 & 0 \\ -\boldsymbol{\psi}^t & 0 & 0 \end{vmatrix}. \tag{71}$$

Finally, because of Eq. (60)

$$\overline{g}=h_0-x_2h_2+O(x_2^2,\eta), \quad \overline{h}=g_0-x_2g_2+O(x_2^2,\eta). \tag{72}$$

The above discussion shows that, up to leading orders in x_2 and η, the original five τ functions f, g, h, $\overline{g}$, $\overline{h}$ can be written in terms of the five τ functions f_0, g_0, g_2, h_0, h_2 depending only on x, x_3, and ξ.

The final part of calculation is to identify an appropriate set of five bilinear equations involving these five τ functions. Applying the reduction to Eqs. (46)–(50) gives

$$D_x^2 g_0\cdot f_0=g_2f_0, \quad D_x^2 h_0\cdot f_0=h_2f_0, \tag{73}$$

$$(D_x^3-4D_{x_3})g_0\cdot f_0=-3D_xg_2\cdot f_0,$$

$$(D_x^3-4D_{x_3})h_0\cdot f_0=-3D_xh_2\cdot f_0, \tag{74}$$

$$D_\xi D_x f_0\cdot f_0=-4g_0h_0. \tag{75}$$

Notice that in this reduction, Eqs. (46) and (47) give Eq. (73); Eqs. (48) and (49) give Eq. (74); and the sum of the equations in Eq. (50) gives Eq. (75).

This set of bilinear equations (73)–(75) is the Hirota form of a $(2+1)$-dimensional Sasa-Satsuma equation. If we define

$$q=\frac{g_0}{f_0}, \quad r=\frac{h_0}{f_0}, \quad q_2=\frac{g_2}{f_0}, \quad r_2=\frac{h_2}{f_0}, \quad \text{and}$$

$$\Phi=\tfrac{1}{2}(\ln f_0)_x, \tag{76}$$

then Eq. (73) gives

$$q_2=q_{xx}+4q\Phi_x, \quad r_2=r_{xx}+4r\Phi_x, \tag{77}$$

Eq. (74) gives

$$q_{xxx}+12q_x\Phi_x-4q_{x_3}=-3q_{2,x},$$

$$r_{xxx}+12r_x\Phi_x-4r_{x_3}=-3r_{2,x}, \tag{78}$$

and Eq. (75) gives

$$\Phi_\xi=-qr. \tag{79}$$

After eliminating q_2 and r_2, one obtains

$$q_{xxx}+6q_x\Phi_x+3q\Phi_{xx}=q_{x_3},$$

$$r_{xxx}+6r_x\Phi_x+3r\Phi_{xx}=r_{x_3},$$

$$\Phi_\xi=-qr. \tag{80}$$

If we now set $r=-q^*$, $x_3=-t$, and use $U=\Phi_x$, we get a $(2+1)$-dimensional Sasa-Satsuma equation

$$q_t+q_{xxx}+6q_xU+3qU_x=0,$$

$$U_\xi=(|q|^2)_x. \tag{81}$$

C. First reduction, step 2

In order to make a dimensional reduction from $(2+1)$- to $(1+1)$-dimensional, we make a further rotation of coordinates

$$x=\tfrac{1}{2}(X+\Xi), \quad \xi=\tfrac{1}{2}(X-\Xi), \tag{82}$$

and then choose eigenfunctions so that the τ functions will be independent of Ξ. Then both X and Ξ derivatives in Eq. (80) become x derivatives and we obtain the Sasa-Satsuma equation with two complex fields

$$q_{xxx}-6q_xqr-3q(qr)_x-q_{x_3}=0,$$

$$r_{xxx}-6r_xqr-3r(qr)_x-r_{x_3}=0. \tag{83}$$

In order to keep the solution structure in this dimensional reduction, it is necessary to choose eigenfunctions $\boldsymbol{\phi}$, $\boldsymbol{\psi}$, and $\boldsymbol{\chi}$ so that they are separable with a common dependence on Ξ. A natural way to achieve this is to take

$$(\phi_0)_i=\lambda_ie^{p_ix+p_i^3x_3}\rightarrow\lambda_ie^{(1/2)p_i\Xi}e^{(1/2)p_iX+p_i^3x_3}, \tag{84}$$

$$\psi_i=\hat{\mu}_ie^{-p_i\xi}\rightarrow\hat{\mu}_ie^{(1/2)p_i\Xi}e^{-(1/2)p_iX}, \tag{85}$$

$$\chi_i=\hat{\nu}_ie^{-p_i\xi}\rightarrow\hat{\nu}_ie^{(1/2)p_i\Xi}e^{-(1/2)p_iX}, \tag{86}$$

where λ_i, $\hat{\mu}_i$, $\hat{\nu}_i$, and p_i are constants. As a result of this choice of eigenfunctions, we have

$$(m_0)_{ij}=c_{ij}+e^{(1/2)p_i\Xi}\left[\frac{\lambda_i\lambda_je^{(1/2)(p_i+p_j)X+(p_i^3+p_j^3)x_3}-(\hat{\mu}_i\hat{\nu}_j+\hat{\nu}_i\hat{\mu}_j)e^{-(1/2)(p_i+p_j)X}}{p_i+p_j}\right]e^{(1/2)p_j\Xi}, \quad \text{for} \quad p_i+p_j\neq0, \tag{87}$$

where c_{ij} are constants of integration [if $p_i+p_j=0$, we must choose the coefficients properly so that from Eq. (41) we get a constant, which can then be absorbed into the c matrix]. As a consequence of the C reduction, the matrix c_{ij} has to be symmetric. In order that $f_0=|m_0|$ be independent of Ξ, we must have

$$\prod_{i=1}^{L} e^{(1/2)p_i\Xi}=\text{const}, \tag{88}$$

and, for each $i,j\in\{1,\ldots,L\}$,

$$c_{ij}e^{-(1/2)(p_i+p_j)\Xi}=\text{const}. \tag{89}$$

These are satisfied if and only if $\Sigma_{i=1}^{L}p_i=0$, and for each $i,j\in\{1,\ldots,L\}$ either $c_{ij}=0$ or $p_i+p_j=0$. Consequently, we take $L=2N$ and then

$$p_{N+i}=-p_i \quad \forall i=1,\ldots,N,$$

$$c_{ij}=\delta_{i+N,j}c_i-\delta_{i,j+N}c_j \quad \forall i,j\in\{1,\ldots,2N\}. \tag{90}$$

Finally, we show how to obtain solutions of the usual Sasa-Satsuma equation (8), in which $r=-q^*$, where $*$ stands for complex conjugation. In order for this to come about, we must have f_0 real and $h_0^*=-g_0$, $h_2^*=-g_2$, and so we must impose the relations

$$\boldsymbol{\phi}_0^*=P\boldsymbol{\phi}_0, \quad \boldsymbol{\psi}^*=P\boldsymbol{\chi}, \quad \boldsymbol{\chi}^*=P\boldsymbol{\psi}, \tag{91}$$

where P is a permutation matrix. The simplest realization of these conditions is to take $N=2M$, choose the permutation to be

$$P=\begin{pmatrix} 0 & \mathbb{I} & 0 & 0\\ \mathbb{I} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbb{I}\\ 0 & 0 & \mathbb{I} & 0\end{pmatrix},$$

with $M\times M$ blocks, and

$$\boldsymbol{\phi}_0=\begin{pmatrix} \lambda_1 e^{(1/2)p_1X+p_1^3x_3}\\ \vdots\\ \lambda_M e^{(1/2)p_MX+p_M^3x_3}\\ \lambda_1^* e^{(1/2)p_1^*X+p_1^{*3}x_3}\\ \vdots\\ \lambda_M^* e^{(1/2)p_M^*X+p_M^{*3}x_3}\\ 0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\end{pmatrix},$$

$$\boldsymbol{\psi}=\begin{pmatrix} 0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\\ \mu_1 e^{(1/2)p_1X}\\ \vdots\\ \mu_M e^{(1/2)p_MX}\\ \nu_1^* e^{(1/2)p_1^*X}\\ \vdots\\ \nu_M^* e^{(1/2)p_M^*X}\end{pmatrix}, \quad \text{and} \quad \boldsymbol{\chi}=\begin{pmatrix} 0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\\ \nu_1 e^{(1/2)p_1X}\\ \vdots\\ \nu_M e^{(1/2)p_MX}\\ \mu_1^* e^{(1/2)p_1^*X}\\ \vdots\\ \mu_M^* e^{(1/2)p_M^*X}\end{pmatrix}, \tag{92}$$

where we have changed the notation for coefficients in order to conform with Eq. (91). Since the $e^{(1/2)p_k\Xi}$ factors in Eqs. (84)–(87) will eventually cancel out with the above choices we do not include them in these formulas, but in order to compensate this omission we must replace $\partial_x^n\phi_0$ by $(2\partial_X)^n\phi_0$, e.g., in Eq. (71).

Taking all constants of integration $c_i=1$ in Eq. (90) gives the M-soliton solution. In particular, the one-soliton solution shown in Eqs. (22)–(25) is obtained for $M=1$, $\lambda_1=1$, $\mu_1=-\rho$, $\nu_1=-\gamma$.

If we set $x_3=-T$ and following Eq. (82) replace D_x and D_ξ with D_X in the bilinear equations (73)–(75) they become

$$D_X^2 g_0\cdot f_0=g_2f_0, \tag{93}$$

$$(D_X^3+4D_T)g_0\cdot f_0=-3D_Xg_2\cdot f_0, \tag{94}$$

$$D_X^2 f_0 \cdot f_0 = 4 g_0 g_0^* . \tag{95}$$

This is the same as Eq. (31) for $\beta=3$, if we identify $f_0 = F$, $g_0 = -G$, and $g_2 = H$. The multisoliton solutions are obtained from Eq. (87), (66), (70), and (71) with Eq. (92). [But please remember that due to the simplified expressions, we have $\partial_x^n \phi_0 = (2\partial_X)^n \phi_0$.]

D. Second reduction, step 1

To obtain the alternative bilinear form of SSNLS, we carry out the reduction process in a different manner. This process will take us via a "coupled Sasa-Satsuma equation" as opposed to the (2+1)-dimensional Sasa-Satsuma equation.

First, we need to introduce two new τ functions

$$s = \begin{vmatrix} m & \boldsymbol{\phi} & \boldsymbol{\phi}_x \\ -\overline{\boldsymbol{\psi}}^t & 0 & 0 \\ -\overline{\boldsymbol{\chi}}^t & 0 & 0 \end{vmatrix}, \quad \overline{s} = \begin{vmatrix} m & \boldsymbol{\psi} & \boldsymbol{\chi} \\ -\overline{\boldsymbol{\phi}}^t & 0 & 0 \\ -\overline{\boldsymbol{\phi}}_x^t & 0 & 0 \end{vmatrix}. \tag{96}$$

In addition to Eqs. (46)–(52), we now have some further bilinear equations satisfied by these τ functions together with the original five τ functions (43)–(45):

$$D_z(D_x^2 - D_{x_2}) g \cdot f = 4 s \overline{h}, \quad D_z(D_x^2 + D_{x_2}) \overline{g} \cdot f = 4 \overline{s} h, \tag{97}$$

$$D_y(D_x^2 - D_{x_2}) h \cdot f = -4 s \overline{g}, \quad D_y(D_x^2 + D_{x_2}) \overline{h} \cdot f = -4 \overline{s} g, \tag{98}$$

$$D_x h \cdot g = s f, \quad D_x \overline{h} \cdot \overline{g} = \overline{s} f. \tag{99}$$

Again these equations are not all independent. Altogether there are seven dependent variables $f, g, \overline{g}, h, \overline{h}, s, \overline{s}$ and two dummy independent variables and, therefore, we need nine independent equations. We can take, e.g., Eqs. (46), (47), (48a), (50), and (99), and then the other equations are consequences of these. [In practice, it is best to keep the full set at one's disposal.]

If we change variables to ξ and η using

$$y = \xi + \eta, \quad z = \xi - \eta, \tag{100}$$

then taking sums and differences of some of the equations, for instance, Eqs. (51a) and (97a), we get some equations containing only ξ derivatives and others containg η derivatives. In the following, we will only use the ones containing ξ derivatives, they are

$$D_\xi(D_x^2 - D_{x_2}) g \cdot f = 4 s \overline{h}, \quad D_\xi(D_x^2 + D_{x_2}) \overline{g} \cdot f = 4 \overline{s} h, \tag{101}$$

$$D_\xi(D_x^2 - D_{x_2}) h \cdot f = -4 s \overline{g}, \quad D_\xi(D_x^2 + D_{x_2}) \overline{h} \cdot f = -4 \overline{s} g, \tag{102}$$

$$D_\xi D_x f \cdot f = -2(g\overline{g} + h\overline{h}). \tag{103}$$

This leaves us with Eqs. (46)–(49), (99), (101)–(103).

At this stage, we have not yet carried out a reduction. If we now do a second change of variables

$$x = \tfrac{1}{2}(X + \Xi), \quad \xi = \tfrac{1}{2}(X - \Xi), \tag{104}$$

we can achieve a dimensional reduction in a manner similar to the dimensional reduction in Sec. III C, i.e., by expanding in η, Ξ and keeping only the leading terms. After also eliminating the x_2 dependence, and denoting $x_3 = -T$, we finally obtain the following set of equations:

$$(D_X^3 + D_T) g \cdot f = 3 s \overline{h}, \quad (D_X^3 + D_T) \overline{g} \cdot f = 3 \overline{s} h, \tag{105}$$

$$(D_X^3 + D_T) h \cdot f = -3 s \overline{g}, \quad (D_X^3 + D_T) \overline{h} \cdot f = -3 \overline{s} g, \tag{106}$$

$$D_X h \cdot g = s f, \quad D_X \overline{h} \cdot \overline{g} = \overline{s} f, \tag{107}$$

$$D_X^2 f \cdot f = -2(g\overline{g} + h\overline{h}). \tag{108}$$

This is a coupled Sasa-Satsuma equation with complex fields. The nonlinear form obtained with the substitutions

$$q = \frac{g}{f}, \quad \overline{q} = \frac{\overline{g}}{f}, \quad r = \frac{h}{f}, \quad \overline{r} = \frac{\overline{h}}{f},$$

is

$$q_T + q_{XXX} - 6 q_X q \overline{q} - 3 \overline{r}(qr)_X = 0,$$

$$\overline{q}_T + \overline{q}_{XXX} - 6 \overline{q}_X \overline{q} q - 3 r(\overline{q}\overline{r})_X = 0,$$

$$r_T + r_{XXX} - 6 r_X r \overline{r} - 3 \overline{q}(rq)_X = 0,$$

$$\overline{r}_T + \overline{r}_{XXX} - 6 \overline{r}_X \overline{r} r - 3 q(\overline{r}\overline{q})_X = 0, \tag{109}$$

which was proposed already in Ref. [13]. If we take $\overline{q} = -q^*, \overline{r} = -r^*$, we obtain Eq. (13).

E. Second reduction, step 2

The final reduction on this system is a reduction of C type, this is obtained as in the other bilinearization by identifying

$$\overline{g} = h, \quad \overline{h} = g, \quad \overline{s} = -s. \tag{110}$$

This gives us the alternative bilinear form of the Sasa-Satsuma equations:

$$(D_X^3 + D_T) g \cdot f = 3 s g, \quad (D_X^3 + D_T) \overline{g} \cdot f = -3 s \overline{g}, \tag{111}$$

$$D_X g \cdot \overline{g} = -s f, \tag{112}$$

$$D_X^2 f \cdot f = -4 g \overline{g}. \tag{113}$$

With $f=F$, $g=-G$, $\overline{g}=G^*$, $s=S$ these equations yield (32) for $\beta=3$. The solutions for this alternative form will be the same as in the first case and the nonlinear form of these equations is Eq. (83), with r replaced by $\overline{q}$. This bilinear form of the system requires a single pure imaginary auxiliary variable s, while the other bilinearization involves a complex auxiliary field h, and consequently, here we need four bilinear equations rather than five.

IV. CONCLUSIONS

In this paper, we have shown how the Sasa-Satsuma equation fits into the general theory as a reduction of the three-component KP hierarchy. As a result, we have obtained two possible bilinearization for SSNLS, (31) and (32), and formulas for constructing multisoliton solutions, (87), (92), (66), (70), (71), and (96). In the reduction process, we have also obtained two intermediate equations, (81) and (109), of which the $(2+1)$-dimensional equation (81) seems to be new.

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